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ScienceDirect
Journal of Sound and Vibration 298 (2006) 1166–1172
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

On the free vibrations of an oscillator with a periodically time-varying mass

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Received 9 February 2006; received in revised form 11 May 2006; accepted 12 June 2006
Available online 4 August 2006

Abstract

In this paper the free vibrations of a linear, single degree of freedom oscillator with a (periodically and stepwise changing) time-varying mass have been studied. Not only solutions of the oscillator equation have been constructed, but also stability diagrams for the free vibrations have been presented for various parameter values.

1. Introduction

In the mechanics of solids and mass-points, numerous applied problems can be found in which material is expelled from a reservoir, or in which material is captured and afterwards transported by some mechanism. Such systems with time-varying mass occur in robotics, rotating crankshafts, conveyor systems, excavators, cranes, biomechanics, and in fluid–structure interaction problems [1,2]. Mechanical systems with a heavy mass and relatively soft spring can be successfully approximated by a single degree of freedom oscillator (sdofo). The oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on their surface are also examples of time-varying dynamic systems [3]. For those constructions the 1-mode Galerkin approximation of the continuous model will lead to a sdofo-equation.

Sdofos are considered as a representative model for testing the numerical behaviour of new computational algorithms with respect to different types of constructions and to the forces which are acting on the system [4]. Especially in the field of aerospace and astronautic engineering, such systems are of interest as they, usually involve a very high computer time.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.06.008

In this paper the free oscillations of a linear sdofo with a (periodically and stepwise changing) time-varying mass will be studied. Not only solutions of this linear oscillator equation will be constructed, but also stability diagrams for the free oscillations will be presented.

Let us consider the oscillations of a sdofo-system with a linear restoring force. The mass of this system is allowed to vary in time according to the periodic stepwise dependence. This model is perhaps the simplest model which describes the process of the vibrations of a cable (cylinder) with rainwater located on it. Raindrops hitting the oscillator may form a water ridge on the oscillator. However, in a stationary situation the mass flow of incoming raindrops hitting the oscillator and the mass flow of raindrops shaken off will be equal. If these mass flows are not equal then the mass of raindrops attached to the oscillator varies in time. Part of the raindrops hitting the cylinder will remain on the surface of the cylinder for some time, and will subsequently be blown or shaken off after some time. It will be assumed when mass is added to or separated from the oscillator that the position of the centre of the (total) mass of the oscillator is not influenced. The following equation of motion for the sdofo can now be derived (see for instance Ref. [1, p. 152]):

$$M\ddot{y} = \dot{M}(w - \dot{y}) - ky + F, \tag{1}$$

where $y = y(t)$ is the displacement of the oscillator (see Fig. 1), $M = M(t)$ is the time-varying mass of the oscillator, $w = w(t)$ is the mean velocity at which masses (i.e. raindrops) are hitting or leaving the oscillator, k is the (positive) stiffness coefficient in the linear restoring force, $F = F(t)$ or $F = F(t, y, \dot{y})$ is an external force, and the dot denotes differentiation with respect to t. An historical overview to obtain Eq. (1) is given in

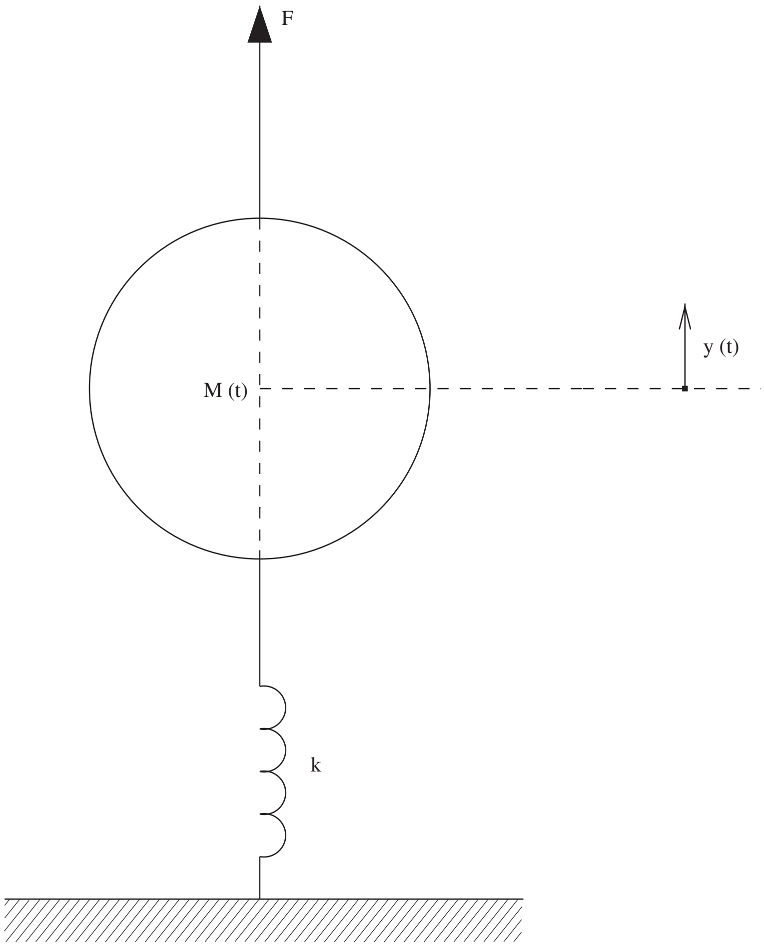

Fig. 1. The single degree of freedom oscillator.

Ref. [1], and goes back to the 18th and 19th century. In the mathematical analysis of Eq. (1) it turns out to be convenient to separate the mass $M(t)$ into a time-invariant part M_0 and into a time-varying part $m(t)$, that is,

$$M(t) = M_0 - m(t), \tag{2}$$

where M_0 is a positive constant, and $M_0 - m(t) > 0$. By substituting Eq. (2) into Eq. (1) it follows that Eq. (1) can be rewritten in:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((M_0 - m(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right) + ky = \frac{-\mathrm{d}m}{\mathrm{d}t}w + F. \tag{3}$$

To study the free vibrations of the oscillator the right-hand side of Eq. (3) should be taken equal to zero (or equivalently take $w = 0$ and $F = 0$). Then, by introducing the time-rescaling $t = \sqrt{(M_0/k)}\tau$ it follows that Eq. (3) becomes

$$((1 - h((\tau))y')' + y = 0, \tag{4}$$

where the prime denotes differentiation with respect to τ, and where $h(\tau) = m(\sqrt{(M_0/k)}\tau)/M_0$ with $1 - h(\tau) > 0$. In this paper it will be assumed that $h(\tau)$ is a periodic step function, that is,

$$h(\tau) = \begin{cases} \varepsilon & \text{for } 0 < \tau < T_0, \\ 0 & \text{for } T_0 < \tau < T \end{cases} \tag{5}$$

and $h(\tau + T) = h(\tau)$, and ε is a constant (in practice usually small) with $0 < \varepsilon < 1$. It should be observed that in the analysis ε is defined to be the quotient m/M_0 , where m is the mass which is added at time T_0, and where M_0 is the mass of the oscillator. So, ε can be seen as a measure for the relative mass which is added time T_0. In Section 2 of this paper an initial value problem for Eq. (4) with $h(\tau)$ given by Eq. (5) will be studied. In particular, stability diagrams will be presented for different values of ε, T_0, and T with $0 < \varepsilon < 1$, and $0 < T_0 < T$. Finally, in Section 3 of this paper some remarks will be made and some conclusions will be drawn.

2. On the stability of the oscillator

In this section, Eq. (4) for $\tau > 0$ will be studied subject to the following initial values at $\tau = 0$:

$$\begin{cases} y(0) = y_0, \\ y'(0) = y_0', \end{cases} \tag{6}$$

where y_0 and y_0' are constants. For $0 < \tau < T_0$ the following equation

$$(1 - \varepsilon)y'' + y = 0 \tag{7}$$

has to be solved subject to Eq. (6). The initial value problem for Eq. (7) can readily be solved, yielding for $0 < \tau < T_0$:

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M}_1(\tau) \quad \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \tag{8}$$

where the (2×2)-matrix $\mathbf{M}_1(\tau)$ is given by

$$\mathbf{M}_1(\tau) = \begin{pmatrix} \cos\left(\dfrac{\tau}{\sqrt{1-\varepsilon}}\right) & \sqrt{1-\varepsilon}\sin\left(\dfrac{\tau}{\sqrt{1-\varepsilon}}\right) \\ \dfrac{-1}{\sqrt{1-\varepsilon}}\sin\left(\dfrac{\tau}{\sqrt{1-\varepsilon}}\right) & \cos\left(\dfrac{\tau}{\sqrt{1-\varepsilon}}\right) \end{pmatrix}. \tag{9}$$

At $\tau = T_0$ a jump discontinuity occurs in the coefficient $h(\tau)$ in (4). The displacement function $y(\tau)$, however, is continuous at $\tau = T_0$. Then, it follows from (4) that also $((1 - h(\tau))y')'$ is continuous at $\tau = T_0$. And so, $(1 - h(\tau))y'$ should be continuous at $\tau = T_0$. So, in the infinitesimal small interval $T_0 - 0 \leqslant \tau \leqslant T_0 + 0$ it follows that:

$$
\begin{aligned}
&(T_0+0) = y(T_0-0),\\
&y'(T_0+0) = (1-\varepsilon)y'(T_0-0),
\end{aligned} \tag{10}
$$

where $y(T_0-0)$ and $y(T_0+0)$ are the limits of $y(T_0)$ when T_0 is approached from the left and from the right side, respectively. For $T_0 - 0 < \tau < T_0 + 0$ Eqs. (10) can be used to obtain:

$$
\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M}_2(\tau)\begin{pmatrix} y(T_0-0) \\ y'(T_0-0) \end{pmatrix} = \mathbf{M}_2(\tau)\mathbf{M}_1(T_0)\begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{11}
$$

where the (2×2)-matrix $\mathbf{M}_2(\tau)$ is given by

$$
\mathbf{M}_2(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}. \tag{12}
$$

For $T_0 < \tau < T$ Eq. (4), given by $y'' + y = 0$, now has to be solved subject to the initial values given at $\tau = T_0 + 0$. This initial value problem can readily be solved, yielding for $T_0 < \tau < T$:

$$
\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M}_3(\tau)\mathbf{M}_2(T_0)\mathbf{M}_1(T_0)\begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{13}
$$

where the (2×2)-matrix $\mathbf{M}_3(\tau)$ is given by

$$
\mathbf{M}_3(\tau) = \begin{pmatrix} \cos(\tau - T_0) & \sin(\tau - T_0) \\ -\sin(\tau - T_0) & \cos(\tau - T_0) \end{pmatrix}. \tag{14}
$$

At $\tau = T$ a jump discontinuity again occurs in the coefficient $h(\tau)$ in Eq. (4). Since the displacement function $y(\tau)$ is continuous at $\tau = T$ it again follows from Eq. (4) that $(1-h(\tau))y'$ should be continuous at $\tau = T$. So, in the infinitesimal small interval $T - 0 \leqslant \tau \leqslant T + 0$ it follows that:

$$
\begin{aligned}
&y(T+0) = y(T-0),\\
&y'(T+0) = \frac{1}{1-\varepsilon}y'(T-0).
\end{aligned} \tag{15}
$$

For $T - 0 < \tau < T + 0$ Eqs. (15) can be used to obtain

$$
\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M}_4(\tau)\mathbf{M}_3(T)\mathbf{M}_2(T_0)\mathbf{M}_1(T_0)\begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{16}
$$

where the (2×2)-matrix $\mathbf{M}_4(\tau)$ is given by

$$
\mathbf{M}_4(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-\varepsilon} \end{pmatrix}.
$$

So far the solution of the initial-value problem for Eq. (4) has been constructed on the interval $0 \leqslant \tau \leqslant T + 0$. To obtain the solution on the interval $nT \leqslant \tau \leqslant (n+1)T + 0$ (with $n = 1, 2, 3, \ldots$) use can again be made of Eqs. (8), (11), (13), and (15), yielding

$$
\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M}_1(\tau - nT).(\mathbf{M}_4(T)\mathbf{M}_3(T)\mathbf{M}_2(T_0)\mathbf{M}_1(T_0))^n\begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}
$$

for $nT < \tau < nT + T_0$, and so on. The stability of the solutions is completely determined by the eigenvalues λ of the matrix

$$
A = \mathbf{M}_4(T)\mathbf{M}_3(T)\mathbf{M}_2(T_0)\mathbf{M}_1(T_0). \tag{17}
$$

If at least one of the moduli of the eigenvalues of matrix $\mathbf{A}$ is larger than 1, then a solution of Eq. (4) can grow exponentially like $\exp((t/T)\ln|\lambda|)$, where λ is an eigenvalue of matrix $\mathbf{A}$ as given by Eq. (17). Matrix $\mathbf{A}$ has the following form:

$$A = \begin{pmatrix} ab - \sqrt{1-\varepsilon}\, cd & \sqrt{1-\varepsilon}\, bc + (1-\varepsilon)\, ad \\ \frac{-1}{1-\varepsilon} ad - \frac{1}{\sqrt{1-\varepsilon}} bc & \frac{-1}{\sqrt{1-\varepsilon}} cd + ab \end{pmatrix}, \tag{18}$$

where

$$a = \cos\left(\frac{T_0}{\sqrt{1-\varepsilon}}\right), b = \cos(T - T_0), c = \sin\left(\frac{T_0}{\sqrt{1-\varepsilon}}\right), \text{ and } d = \sin(T - T_0). \tag{19}$$

Matrix $\mathbf{A}$ has the following properties:

$$\begin{aligned} \det(\mathbf{A}) =& \det(\mathbf{M}_4(T))\det(\mathbf{M}_3(T))\det(\mathbf{M}_2(T_0))\det(\mathbf{M}_1(T_0)) \\ &= \frac{1}{1-\varepsilon}.1.1.(1-\varepsilon) = 1, \\ \mathrm{tr}(\mathbf{A}) &= 2ab - \frac{(2-\varepsilon)}{\sqrt{1-\varepsilon}}\, cd, \end{aligned} \tag{20}$$

where $\det(\mathbf{A})$ is the determinant of $\mathbf{A}$, and $\mathrm{tr}(\mathbf{A})$ is the trace of A, respectively, and where a, b, c, and d are given by Eq. (19). The eigenvalues $\lambda_{1,2}$ of matrix $\mathbf{A}$ are given by (using Eq. (20)):

$$\lambda_{1,2} = \tfrac{1}{2}\mathrm{tr}(\mathbf{A}) \pm \tfrac{1}{2}\sqrt{D}, \tag{21}$$

where

$$D = (\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) = (\mathrm{tr}(\mathbf{A}))^2 - 4. \tag{22}$$

To determine the stability of the solutions three cases have to be considered: $D<0, D>0$, and $D=0$. For $D<0$ or equivalently for $-2<\mathrm{tr}(\mathbf{A})<2$ it follows from Eq. (21) that $\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) \pm (i/2)\sqrt{-D}$, implying (using Eq. (22))

$$|\lambda_{1,2}|^2 = \tfrac{1}{4}(\mathrm{tr}(\mathbf{A}))^2 - \tfrac{1}{4}D = 1.$$

So, for $D<0$ the free vibrations of the oscillator are stable (that is, there is no exponential or linear growth in time of the oscillations). For $D>0$ or equivalently for $\mathrm{tr}(\mathbf{A})< -2$ or $\mathrm{tr}(\mathbf{A})>2$ it follows from Eq. (21) that one of the eigenvalues $\lambda_{1,2}$ has a modulus which is larger than 1. So for $D>0$ the free vibrations of the oscillator are unstable (that is, there is exponential growth in time of the oscillations). For $D=0$ or equivalently for $\mathrm{tr}(\mathbf{A}) = 2$ or $\mathrm{tr}(\mathbf{A}) = -2$ it follows from Eq. (21) that $\lambda = 1$ when $\mathrm{tr}(\mathbf{A}) = 2$, and $\lambda = -1$ when $\mathrm{tr}(\mathbf{A}) = -2$. In both cases the multiplicity of the eigenvalues is two. When the dimension of the eigenspace belonging to the eigenvalue $\lambda = 1$ (or $\lambda = -1$) is equal to two the free vibrations of the oscillator will be stable else the vibrations will grow linearly in time. The dimension of the eigenspace belonging to the eigenvalue $\lambda = 1$ (or $\lambda = -1$) with multiplicity two is two if and only if $A - \lambda I = O$, where I is the (2×2) identity matrix, and O the (2×2) zero matrix. For $\lambda = 1$ it follows that $A - I = O$ is equivalent with the system

$$\begin{aligned} &ab - \sqrt{1-\varepsilon}\, cd - 1 = 0, \\ &\sqrt{1-\varepsilon} bc + (1-\varepsilon)\, ad = 0, \\ &\frac{-1}{1-\varepsilon} ad - \frac{1}{\sqrt{1-\varepsilon}} bc = 0, \\ &\frac{-1}{\sqrt{1-\varepsilon}} cd + ab - 1 = 0, \end{aligned} \tag{23}$$

where a, b, c, and d are given by Eq. (19). The second and the third equation in system (23) can be rewritten in:

$$\begin{pmatrix} \sqrt{1-\varepsilon} & 1-\varepsilon \\ \frac{-1}{\sqrt{1-\varepsilon}} & \frac{-1}{1-\varepsilon} \end{pmatrix} \begin{pmatrix} bc \\ ad \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{24}$$

Since the determinant of the coefficient-matrix in Eq. (24) is equal to $-\varepsilon/\sqrt{1-\varepsilon}$, which is nonzero for $0<\varepsilon<1$, it follows from Eq. (24) that $bc=0$ and $ad=0$. Using Eq. (19) it then follows that $a=b=0$ or $c=d=0$. When $a=b=0$ it follows from the first and the last equation in Eq. (23) that $cd=-1/\sqrt{1-\varepsilon}$ and $cd=-\sqrt{1-\varepsilon}$, which is impossible for $0<\varepsilon<1$. When $c=d=0$ it follows from Eq. (23) that $ab=1$. So, for $\lambda=1$, that is, for $\mathrm{tr}(\mathbf{A})=2$ the dimension of the eigenspace for the eigenvalue $\lambda=1$ is only equal to two when (using Eq. (19)):

$$\begin{cases} e=0 \\ d=0 \\ ab=1 \end{cases} \Rightarrow \begin{cases} \sin\left(\dfrac{T_0}{\sqrt{1-\varepsilon}}\right)=0 \\ \sin(T-T_0)=0 \\ \cos\left(\dfrac{T_0}{\sqrt{1-\varepsilon}}\right)\cos(T-T_0)=1 \end{cases} \Rightarrow \begin{cases} T_0=n_1\pi\sqrt{1-\varepsilon} \\ T_1-T_0=n_2\pi \\ (-1)^{n_1}.(-1)^{n2}=1 \end{cases} \tag{25}$$

with $n_1\epsilon\mathbb{N}^+$, $n_2\epsilon\mathbb{N}^+$, and n_1+n_2 is even. For $\lambda=-1$ a completely similar analysis can be given (replace in the first and in the last equation of Eq. (23) -1 by $+1$, and replace in Eq. (25) $ab=1$ by $ab=-1$), yielding Eq. (25) but now with n_1+n_2 is odd. So, for $D=0$, and $T_0=n_1\pi\sqrt{1-\varepsilon}$ and $T_1=T_0+n_2\pi$ with $n_1,n_2\epsilon\mathbb{N}^+$

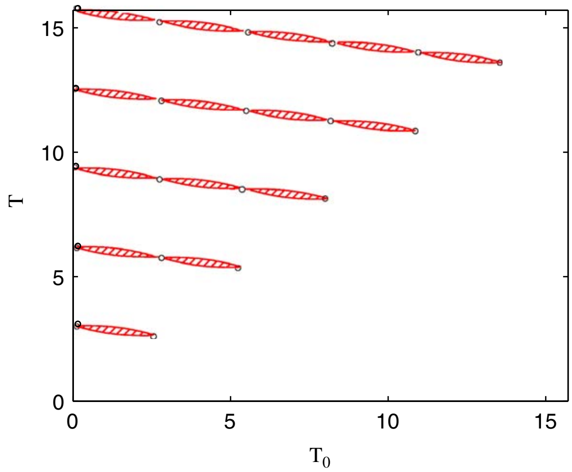

Fig. 2. Stability and instablity regions for the free vibrations of the oscillator with $\varepsilon=0.25$ and $0\leqslant T_0\leqslant T\leqslant 5\pi$ (grey colouring: unstable; white colouring: stable; continuous lines: unstable; o points: stable).

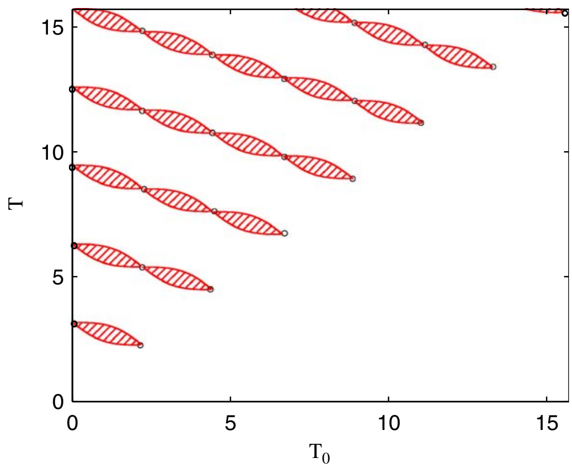

Fig. 3. Stability and instability regions for the free vibrations of the oscillator with $\varepsilon=0.5$ and $0\leqslant T_0\leqslant T\leqslant 5\pi$ (grey colouring: unstable; white colouring: stable; continuous lines: unstable; o points: stable).

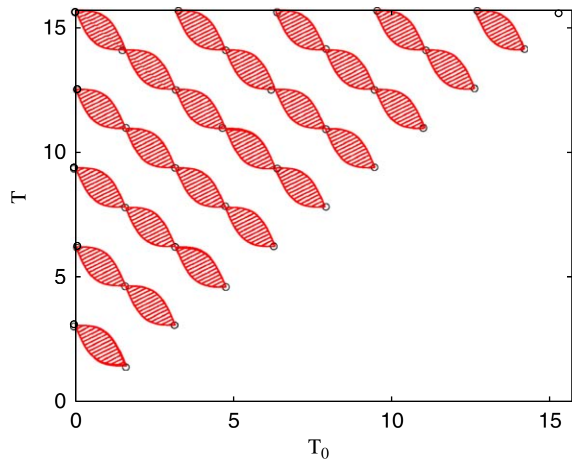

Fig. 4. Stability and instablity regions for the free vibrations of the oscillator with $\varepsilon = 0.75$ and $0 \leqslant T_0 \leqslant T \leqslant 5\pi$ (grey colouring: unstable; white colouring: stable; continuous lines: unstable; o points: stable).

the free vibrations of the oscillator will be stable. In all other cases for $D = 0$ the free vibrations will be unstable (that is, the oscillations will grow linearly in time). So far, it can be concluded that for $\mathrm{tr}(\mathbf{A}) \geqslant 2$ and for $\mathrm{tr}(\mathbf{A}) \leqslant -2$ (except for those values of T and T_0 for which $D = 0$, $T_0 = n_1\pi\sqrt{1-\varepsilon}$, and $T_1 = T_0 + n_2\pi$ with $n_1, n_2 \epsilon \mathbb{N}^+$) the free vibrations of the oscillator are unstable, and that for $2 < \mathrm{tr}(\mathbf{A}) < 2$ the free oscillations are stable. So, the boundaries of the (in-)stability regions are give by $\mathrm{tr}(\mathbf{A}) = 2$ and $\mathrm{tr}(\mathbf{A}) = -2$, where $\mathrm{tr}(\mathbf{A}) = 2\cos(T_0/\sqrt{1-\varepsilon})\cos(T - T_0) - (2-\varepsilon)/\sqrt{1-\varepsilon}\sin(T_0/\sqrt{1-\varepsilon})\sin(T - T_0)$. In Figs. 2–4 the instability regions (indicated by a grey colouring) and the stability region (indicated by a white colouring) in the (T_0, T)-plane are given for $\varepsilon = 0.25$, 0.5, and 0.75, respectively. From Figs. 2–4 it is clear that for larger values of ε the instability regions also become larger.

3. Conclusions and remarks

In this paper the stability of the free vibrations of a linear, single degree of freedom oscillator with a periodically and stepwise changing time-varying mass has been studied. By adding a mass m at time T_0 to the oscillator with mass M_0, by removing this amount of mass at time $T > T_0$, and by repeating this procedure with period T, it has been shown mathematically that for certain intervals of the parameter values $\varepsilon = m/M_0$, T_0, and T the free vibrations of the oscillator can be unstable. In the presented analysis it has been assumed that $0 < \varepsilon = m/M_0 < 1$, that is, it has been assumed that the added mass m at time T_0 is smaller than the mass M_0 of the oscillator. This instablity mechanism may serve as a very simple (sub-) model to describe certain aspects of instability for the rain-wind induced oscillations of elastic structures such as cables in windfields with water rivulets on the cable-surfaces. To obtain more realistic models periodically and multi-stepwise changing time-varying masses can be considered. Also the velocity at which masses are hitting or leaving the oscillator can be taken into account (see Eq. (3). Other external forces (such as drag- and lift forces, damping forces, and so on) can be included in the model equation (3). The aforementioned extensions to the model equation (4) can be interesting subjects for future research.

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